

九十六學年第一學期 PHYS2310 電磁學 期中考試題(共兩頁)

[Griffiths Ch.1-3] 2007/11/13, 10:10am-12:00am, 教師：張存續

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Useful formulas

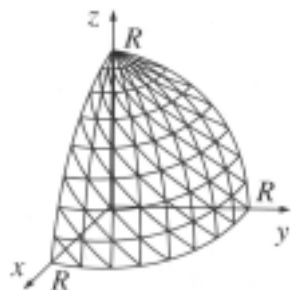
$$\nabla V = \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} \quad \text{and} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi$$

1. (8%,12%) $\mathbf{v} = r^2 \cos \theta \hat{\mathbf{r}} + r^2 \cos \phi \hat{\boldsymbol{\theta}} - r^2 \cos \theta \sin \phi \hat{\boldsymbol{\phi}}$

(a) Compute $\nabla \cdot \mathbf{v}$.

(b) Check the divergence theorem using the volume shown in the figure (one octant of the sphere of radius R).

[Hint: Make sure you include the entire surface.]



2. (6%,7%,7%) A solid sphere of radius R is uniformly charged with the volume charge density ρ .

(a) Find the voltage V and the electric field $\mathbf{E}$ in the two regions $r \leq R$ and $r > R$.

(b) Find the total energy using ρ and V .

(c) Find the total energy using $\mathbf{E}$ and V . Take a spherical volume of radius a ($a > R$).

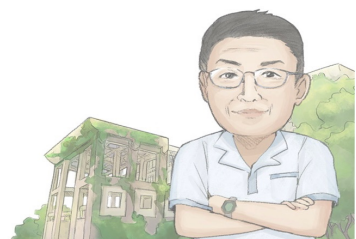
[Hint: The energy $W = \frac{1}{2} \int \rho V d\tau = \frac{\epsilon_0}{2} \left(\int_V E^2 d\tau + \oint_S V \mathbf{E} \cdot d\mathbf{a} \right).$]

3. (10%, 10%) A metal sphere of radius R carries a total charge Q .

(a) Use the boundary condition to determine the surface charge density σ on the metal sphere ($r = R$).

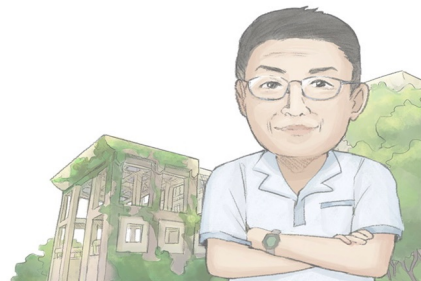
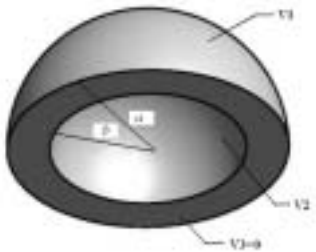
(b) Calculate the force of repulsion between the “northern” hemisphere and the “southern” hemisphere.

[Hint: The repulsive force per unit area (or the electrostatic pressure) $\mathbf{f} = \frac{\epsilon_0}{2} E^2 \hat{\mathbf{n}}.$]



4. (7%,7%,6%) A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane.
- Find the potential V in the region above the plane.
 - Find the surface charge density σ induced on the conducting plane.
 - Find the force on the wire per unit length.
- [Hint: Use the method of images.]

5. (6%, 8%, 6%) Suppose the potential at the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a, \theta) = V_0(5\cos^3 \theta - 3\cos \theta)$, $V_2(b, \theta) = 0$, $V_3(r, \pi/2) = 0$. V_0 is a constant.
- Show the general solution in the region $b \leq r \leq a$.
 - Determine the potential in the region $b \leq r \leq a$, using the boundary conditions.
 - Calculate the electric field at the inner shell $\mathbf{E}(r = b)$ and outer shell $\mathbf{E}(r = a)$.
- [Hint: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, and $P_3(x) = (5x^3 - 3x)/2$.]



1.

(a)

$$\mathbf{v} = r^2 \cos \theta \hat{\mathbf{r}} + r^2 \cos \phi \hat{\boldsymbol{\theta}} - r^2 \cos \theta \sin \phi \hat{\boldsymbol{\phi}}$$

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^2 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta r^2 \cos \phi) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (-r^2 \cos \theta \sin \phi) \\ &= 4r \cos \theta + r \frac{\cos \theta}{\sin \theta} \cos \phi - r \frac{\cos \theta}{\sin \theta} \cos \phi \\ &= 4r \cos \theta \end{aligned}$$

(b)

The divergence theorem $\int_V \nabla \cdot \mathbf{v} d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}$

$$\begin{aligned} \int_V \nabla \cdot \mathbf{v} d\tau &= \int_0^R \int_0^{\pi/2} \int_0^{\pi/2} (4r \cos \theta) r^2 \sin \theta dr d\theta d\phi = \frac{\pi}{2} \int_0^R \int_0^{\pi/2} (4r^3 \cos \theta \sin \theta) dr d\theta \\ &= \frac{\pi}{16} \int_0^R \int_0^{\pi/2} (4 \sin 2\theta) dr^4 d\theta = \frac{\pi}{4} R^4 \int_0^{\pi/2} \sin 2\theta d2\theta = \frac{\pi}{4} R^4 \end{aligned}$$

$\oint_S \mathbf{v} \cdot d\mathbf{a}$ = xy-plane + yz-plane + zx-plane + curved surface

xy-plane: $d\mathbf{a} = -r dr d\theta \hat{\boldsymbol{\phi}}$, $\phi = 0$, $\mathbf{v} \cdot d\mathbf{a} = (r^2 \cos \theta \sin \phi) r dr d\theta = 0$,

yz-plane: $d\mathbf{a} = r dr d\theta \hat{\boldsymbol{\phi}}$, $\phi = \pi/2$, $\mathbf{v} \cdot d\mathbf{a} = -(r^2 \cos \theta \sin \phi) r dr d\theta = -r^3 \cos \theta dr d\theta$, $\int_0^R \int_0^{\pi/2} -(r^3 \cos \theta) dr d\theta = -\frac{1}{4} R^4$

zx-plane: $d\mathbf{a} = r dr d\phi \hat{\boldsymbol{\theta}}$, $\theta = \pi/2$, $\mathbf{v} \cdot d\mathbf{a} = (r^2 \cos \phi) r dr d\phi = r^3 \cos \phi dr d\phi$, $\int_0^R \int_0^{\pi/2} (r^3 \cos \phi) dr d\phi = \frac{1}{4} R^4$

curved surface: $d\mathbf{a} = R^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}$, $r = R$, $\mathbf{v} \cdot d\mathbf{a} = (R^2 \cos \theta) R^2 \sin \theta d\theta d\phi = \frac{R^4}{2} \sin 2\theta d\theta d\phi$,

$$\int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{R^4}{2} \sin 2\theta \right) d\theta d\phi = \frac{\pi}{4} R^4$$

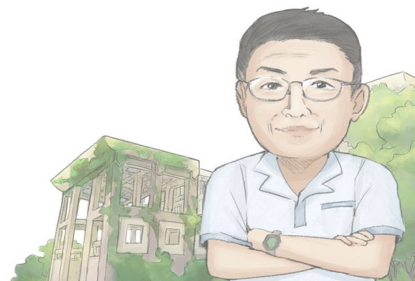
$$\oint_S \mathbf{v} \cdot d\mathbf{a} = 0 - \frac{1}{4} R^4 + \frac{1}{4} R^4 + \frac{\pi}{4} R^4 = \frac{\pi}{4} R^4 = \int_V \nabla \cdot \mathbf{v} d\tau$$

2.

(a)

Use Gauss's law, we obtain $E = \begin{cases} \frac{1}{4\pi\epsilon_0 r^2} \frac{4}{3} \pi \rho r^3 = \frac{\rho}{3\epsilon_0} r, & \text{for } r \leq R \\ \frac{1}{4\pi\epsilon_0 r^2} \frac{4}{3} \pi \rho R^3 = \frac{\rho R^3}{3\epsilon_0} \frac{1}{r^2}, & \text{for } r \geq R \end{cases}$ in $\hat{\mathbf{r}}$

$$V = \begin{cases} -\int_{\infty}^R \mathbf{E} \cdot d\mathbf{l} - \int_R^r \mathbf{E} \cdot d\mathbf{l} = \frac{\rho R^3}{3\epsilon_0} \frac{1}{R} + \frac{\rho}{6\epsilon_0} (R^2 - r^2) = \frac{\rho}{6\epsilon_0} (3R^2 - r^2), & \text{for } r \leq R \\ -\int_{\infty}^r \mathbf{E} \cdot d\mathbf{l} = -\int_{\infty}^r \frac{\rho R^3}{3\epsilon_0} \frac{1}{r^2} \cdot dr = \frac{\rho R^3}{3\epsilon_0} \frac{1}{r}, & \text{for } r \geq R \end{cases}$$



$$(b) \quad W = \frac{1}{2} \int \rho V d\tau = \frac{\rho}{2} \int_0^R V d\tau$$

$$\begin{aligned} W &= \frac{1}{2} \int \rho V d\tau = \frac{\rho}{2} \int_0^R \left(\frac{\rho}{6\epsilon_0} (3R^2 - r^2) \right) r^2 dr \iint \sin \theta d\theta d\phi \\ &= \frac{\rho}{2} \frac{\rho}{6\epsilon_0} \left(3R^2 \frac{R^3}{3} - \frac{R^5}{5} \right) 4\pi \\ &= \frac{4\pi \rho^2 R^5}{15\epsilon_0} \end{aligned}$$

(c)

$$E^2 = \begin{cases} \frac{\rho^2}{9\epsilon_0^2} r^2, & \text{for } r \leq R \\ \frac{\rho^2}{9\epsilon_0^2} \frac{R^6}{r^4}, & \text{for } r \geq R \end{cases} \quad \text{and} \quad V\mathbf{E} = \frac{\rho R^3}{3\epsilon_0} \frac{1}{r^2} \frac{\rho R^3}{3\epsilon_0} \frac{1}{r} \hat{\mathbf{r}} = \frac{\rho^2}{9\epsilon_0^2} \frac{R^6}{r^3} \hat{\mathbf{r}}$$

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \left(\int_V E^2 d\tau + \oint_S V\mathbf{E} \cdot d\mathbf{a} \right) = \frac{\epsilon_0}{2} \left(\int_0^R E^2 d\tau + \int_R^a E^2 d\tau + \oint_{r=a} V\mathbf{E} \cdot d\mathbf{a} \right) \\ &= \frac{\epsilon_0}{2} \left(\frac{\rho^2}{9\epsilon_0^2} \frac{R^5}{5} - \frac{\rho^2}{9\epsilon_0^2} \left(\frac{R^6}{a} - \frac{R^6}{R} \right) \right) 4\pi + \frac{\epsilon_0}{2} \left(\frac{\rho^2}{9\epsilon_0^2} \frac{R^6}{a} \right) 4\pi \\ &= \frac{\epsilon_0}{2} \left(\frac{\rho^2}{9\epsilon_0^2} \frac{6R^5}{5} \right) 4\pi + \frac{\epsilon_0}{2} \left(\frac{\rho^2}{9\epsilon_0^2} \left(\frac{R^6}{a} - \frac{R^6}{a} \right) \right) 4\pi \\ &= \frac{4\pi \rho^2 R^5}{15\epsilon_0} \end{aligned}$$

3.

(a)

$$\text{Simple solution: } \sigma = \frac{Q}{A} = \frac{Q}{4\pi R^2}$$

$$\text{Use Gauss's law, we obtain } E = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{\mathbf{r}} \quad \text{for } r > R$$

$$\text{Normal component } (E_{\text{above}} - E_{\text{below}}) = \frac{\sigma}{\epsilon_0}, \quad E_{\text{below}} = 0, \quad \sigma = \frac{Q}{4\pi R^2}$$

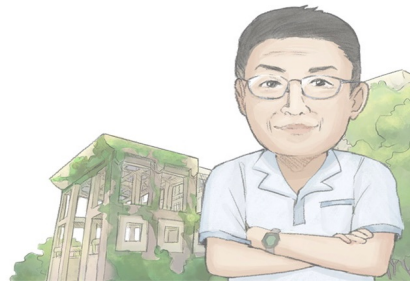
(b)

$$\text{Use Gauss's law, we obtain } E = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{\mathbf{r}} \quad \text{for } r > R$$

$$\mathbf{f} = \epsilon_0 E^2 / 2 \hat{\mathbf{n}} = E = \frac{Q^2}{32\pi^2 \epsilon_0} \frac{1}{r^4} \hat{\mathbf{r}}$$

$$\text{Only z-component of } \mathbf{f} \text{ will survive } f_z = \frac{Q^2}{32\pi^2 \epsilon_0} \frac{1}{R^4} \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \frac{Q^2}{32\pi^2 \epsilon_0} \frac{1}{R^4} \cos \theta$$

$$F_z = \int_0^{\pi/2} \int_0^\pi f_z R^2 \sin \theta d\theta d\phi = \frac{Q^2}{32\pi^2 \epsilon_0} \frac{2\pi}{R^2} \int_0^{\pi/2} \cos \theta \sin \theta d\theta = \frac{Q^2}{32\pi \epsilon_0} \frac{1}{R^2}$$



4.

(a)

Assume the image line charge of $-\lambda$ is placed at a distance d below the plane.

Using the Gauss's law, the electric field outside a line charge λ is $\mathbf{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} \hat{\mathbf{r}}$.

$$\text{So } V = -\int_{r_0}^r \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r} = V(r) - V_{ref}(r_0)$$

$$V = V_+ + V_- = \frac{\lambda}{2\pi\epsilon_0} \left(\ln \frac{r_0}{\sqrt{(x-d)^2 + y^2}} - \ln \frac{r_0}{\sqrt{(x+d)^2 + y^2}} \right) = \frac{\lambda}{4\pi\epsilon_0} \left(\ln \frac{(x+d)^2 + y^2}{(x-d)^2 + y^2} \right)$$

(b)

$$\begin{aligned} \sigma &= \epsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}} = \epsilon_0 E_x = -\frac{\partial}{\partial x} \frac{\lambda}{4\pi} \left(\ln \frac{(x+d)^2 + y^2}{(x-d)^2 + y^2} \right) \bigg|_{x=0} = -\frac{\lambda}{4\pi} \left(\frac{2(x+d)}{(x+d)^2 + y^2} - \frac{2(x-d)}{(x-d)^2 + y^2} \right) \bigg|_{x=0} \\ &= -\frac{\lambda}{4\pi} \frac{4d}{d^2 + y^2} = -\frac{\lambda}{\pi} \frac{d}{d^2 + y^2} \end{aligned}$$

$$\text{Simple check: } \lambda' = \int_{-\infty}^{\infty} \sigma dy = \int_{-\infty}^{\infty} -\frac{\lambda}{\pi} \frac{d}{d^2 + y^2} dy$$

$$\text{Let } y = d \tan \theta, dy = d \sec^2 \theta d\theta$$

$$\lambda' = -\frac{\lambda}{\pi} \int_{-\pi/2}^{\pi/2} \frac{d^2 \sec^2 \theta}{d^2 \sec^2 \theta} d\theta = -\lambda$$

(c)

$$dF = Edq = E\lambda d\ell$$

$$\frac{dF}{d\ell} = E\lambda = \frac{\lambda}{2\pi\epsilon_0(2d)} \lambda = \frac{\lambda^2}{4\pi\epsilon_0 d}$$

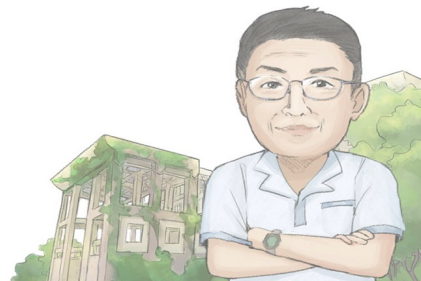
5.

(a)

$$\text{Boundary condition } \begin{cases} \text{(i) } V_1(a, \theta) = V_0(5 \cos^3 \theta - 3 \cos \theta) \\ \text{(ii) } V_2(b, \theta) = 0 \\ \text{(ii) } V_3(r, \theta = \pi/2) = 0 \end{cases}$$

$$\text{General solution } V(r, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-(\ell+1)}) P_{\ell}(\cos \theta)$$

(b)



$$\text{B.C. (ii)} \rightarrow V(b, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} b^{\ell} + B_{\ell} b^{-(\ell+1)}) P_{\ell}(\cos \theta) = 0 \Rightarrow B_{\ell} = -A_{\ell} b^{2\ell+1}$$

$$\text{B.C. (i)} \rightarrow V(a, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} b^{\ell} + B_{\ell} b^{-(\ell+1)}) P_{\ell}(\cos \theta) = 2V_0 P_3(\cos \theta)$$

Comparing the coefficient $\Rightarrow A_3 a^3 + B_3 a^{-4} = 2V_0$, $A_{\ell} = B_{\ell} = 0$ for $\ell = 0, 1, 2, 4, 5, \dots$

$$\text{B.C. (iii)} \rightarrow V(r, \theta = \frac{\pi}{2}) = (A_3 r^3 + B_3 r^{-3}) P_3(0) = 0$$

$\Rightarrow A_{\ell} = B_{\ell} = 0$ except $\ell = 3$,

$$A_3 = \frac{2V_0 a^4}{a^7 - b^7} \text{ and } B_3 = -\frac{2V_0 a^4 b^7}{a^7 - b^7}$$

$$\therefore V(r, \theta) = \left(\frac{V_0}{a^7 - b^7} a^4 r^3 - \frac{V_0}{a^7 - b^7} a^4 b^7 r^{-4} \right) (5 \cos^3 \theta - 3 \cos \theta)$$

(c)

$$V(r, \theta) = \left(\frac{V_0}{a^7 - b^7} a^4 r^3 - \frac{V_0}{a^7 - b^7} a^4 b^7 r^{-4} \right) (5 \cos^3 \theta - 3 \cos \theta)$$

$$\begin{aligned} \mathbf{E} &= -\nabla V = -\frac{\partial V}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} \\ &= -\left(\frac{3V_0}{a^7 - b^7} a^4 r^2 + \frac{4V_0}{a^7 - b^7} a^4 b^7 r^{-5} \right) (5 \cos^3 \theta - 3 \cos \theta) \hat{\mathbf{r}} \\ &\quad + \left(\frac{V_0}{a^7 - b^7} a^4 r^3 - \frac{V_0}{a^7 - b^7} a^4 b^7 r^{-4} \right) (15 \cos^2 \theta \sin \theta - 3 \sin \theta) \hat{\boldsymbol{\theta}} \end{aligned}$$

$$\mathbf{E}(r = a) = -\frac{V_0}{a} \left(\frac{3a^7 + 4b^7}{a^7 - b^7} \right) (5 \cos^3 \theta - 3 \cos \theta) \hat{\mathbf{r}} + V_0 \left(\frac{1}{a} \right) (15 \cos^2 \theta \sin \theta - 3 \sin \theta) \hat{\boldsymbol{\theta}}$$

$$\mathbf{E}(r = b) = -V_0 \left(\frac{7a^4 b^2}{a^7 - b^7} \right) (5 \cos^3 \theta - 3 \cos \theta) \hat{\mathbf{r}}$$

